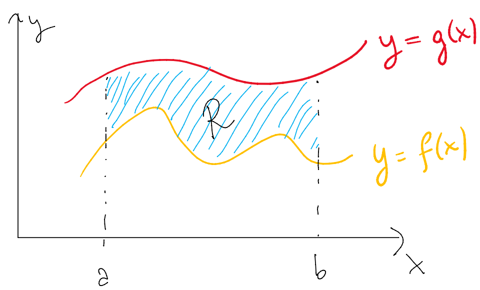
Section 5.7

Monday, April 20, 2020 5:54 PM

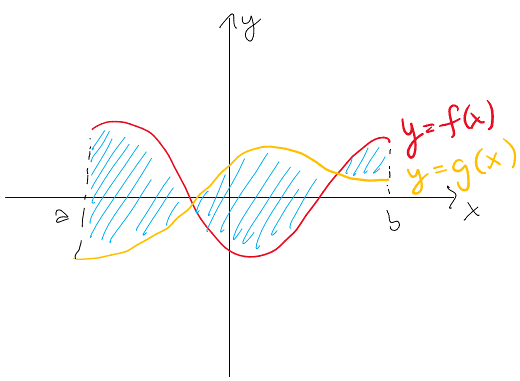
Areas of plane regions

Let f and g be continuous functions over an interval $[a,b]$ with $f(x) \le g(x)$ for all x in $[a,b]$. Then the area of the region R between the graphs of $f(x)$ and $g(x)$ over $[a,b]$ is



$$\int_a^b \big(\underbrace{g(x)}_{\text{"top" function}} - \underbrace{f(x)}_{\text{"bottom" function}}\big)\,dx = \text{the area of } R$$

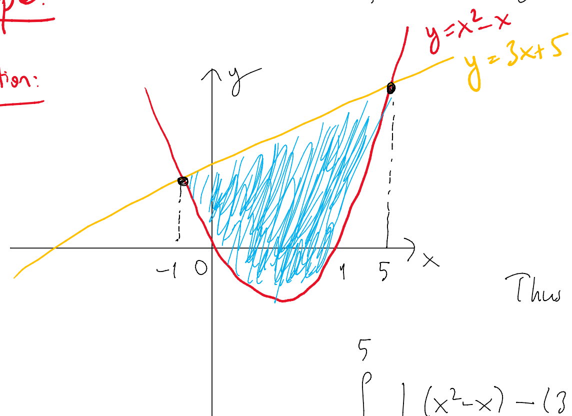
In general, even if we do not have $f(x) \le g(x)$ or $g(x) \le f(x)$ over $[a,b]$, we can find the area of the region between $y=f(x)$ and $y=g(x)$ over $[a,b]$ by the integral



$$\int_a^b |f(x) - g(x)|\,dx = \text{the area of the blue region}$$

Example: Find the area of the region in plane bounded by $y=x^2-x$ and $y=3x+5$.

Solution:



We first find the intersection points of these curves.

$$\left.\begin{aligned} y &= 3x+5 \\ y &= x^2-x \end{aligned}\right\} \quad \begin{aligned} x^2-x &= 3x+5 \\ x^2-4x-5 &= 0 \\ (x-5)(x+1) &= 0 \end{aligned} \quad \Rightarrow \quad x=5 \text{ or } x=-1$$

Thus the area of this region is

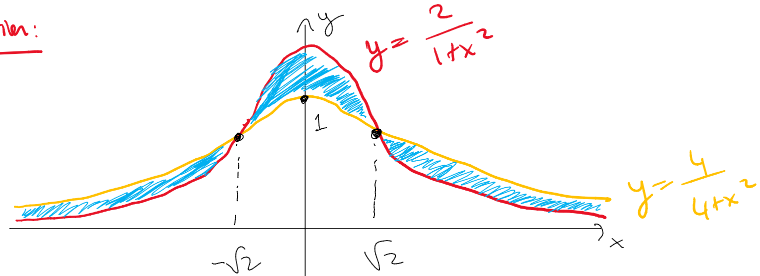
$$\int_{-1}^{5} |(x^2-x)-(3x+5)|\,dx = \int_{-1}^{5} (3x+5)-(x^2-x)\,dx$$

$$= \left(\frac{3x^2}{2} + 5x - \frac{x^3}{3} + \frac{x^2}{2} \right) \Big|_{-1}^{5} = \dots$$

Remark: One is allowed have $a = -\infty$ and/or $b = +\infty$ in the remarks above. Similarly, these remarks hold even if $\int_a^b f(x)\,dx$ and/or $\int_a^b g(x)\,dx$ are improper and f and g are continuous only on (a,b).

Example: Find the area of the region between $y = \frac{2}{1+x^2}$ and $y = \frac{4}{4+x^2}$.

Solution:



The area of the region between these curves is

$$\int_{-\infty}^{\infty} \left| \frac{2}{1+x^2} - \frac{4}{4+x^2} \right| dx$$

Exercise: Using the techniques that we learned before, double check that the graphs indeed look like this!

In order to evaluate this integral, we need to find find the intersection points.

$$\left. \begin{array}{l} y = \dfrac{2}{1+x^2} \\ y = \dfrac{4}{4+x^2} \end{array} \right\} \quad \frac{2}{1+x^2} = \frac{4}{4+x^2} \Rightarrow x = \mp\sqrt{2}$$
$$8 + 2x^2 = 4 + 4x^2$$
$$4 = 2x^2$$

So we have

$$\int_{-\infty}^{\infty} \left| \frac{2}{1+x^2} - \frac{4}{4+x^2} \right| dx =$$

$$\int_{-\infty}^{-\sqrt{2}} \left(\frac{4}{4+x^2} - \frac{2}{1+x^2} \right) dx + \int_{-\sqrt{2}}^{\sqrt{2}} \left(\frac{2}{1+x^2} - \frac{4}{4+x^2} \right) dx + \int_{\sqrt{2}}^{\infty} \left(\frac{4}{4+x^2} - \frac{2}{1+x^2} \right) dx =$$

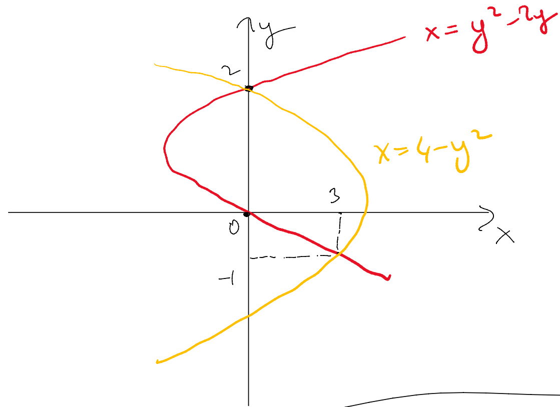
... EXERCISE!

$$\boxed{\begin{array}{l} \displaystyle\int \frac{4}{4+x^2}\,dx = 2\arctan\left(\frac{x}{2}\right) + C \\[2ex] \displaystyle\int \frac{2}{1+x^2}\,dx = 2\arctan(x) + C \end{array}}$$

Remark: Sometimes it is convenient to interchange the roles of x and y, that is, see

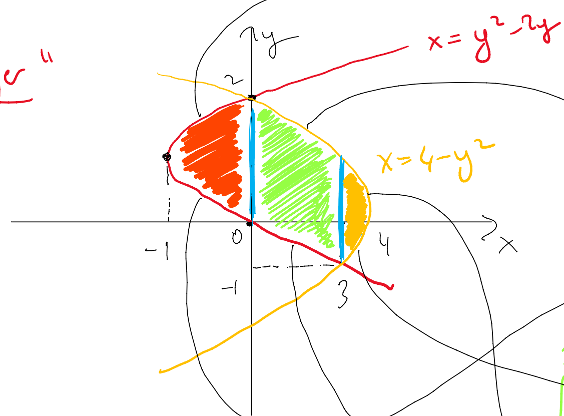
the region as a region between the graphs of functions of y over an interval for y.
In this case, we proceed as before after changing the roles of x and y in the integrals in the above remarks.

Example: Find the area of the region bounded by $x=y^2-2y$ and $x=4-y^2$.



$$\left.\begin{aligned} x&=y^2-2y \\ x&=4-y^2 \end{aligned}\right\} \quad \begin{aligned} y^2-2y&=4-y^2 \\ 2y^2-2y&=4 \\ y^2-y-2&=0 \\ (y-2)(y+1)&=0 \end{aligned} \quad \Rightarrow \quad \begin{aligned} &y=2 \\ &\text{or} \\ &y=-1 \end{aligned}$$

The "longer" solution:



the area of the orange region $=$

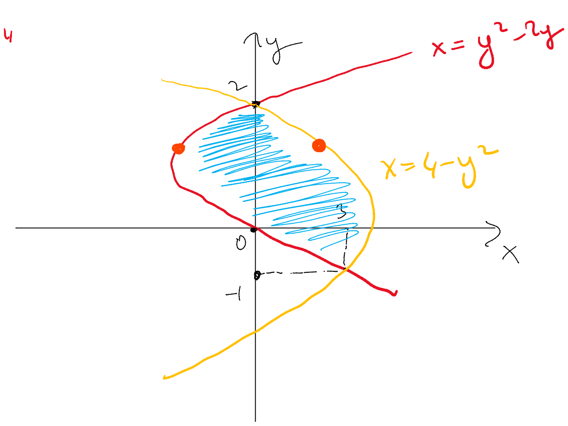
$$\int_{-1}^{0} (1+\sqrt{x+1})-(1-\sqrt{x+1})\,dx$$

the area of the green region $=$

$$\int_{0}^{3} \sqrt{4-x}-(1-\sqrt{x+1})\,dx$$

the area of the yellow region $= \int_{3}^{4} \left(\sqrt{4-x}-\left(-\sqrt{4-x}\right)\right)dx$

The "easier" solution:



The blue region is between the graphs of functions $g(y)=4-y^2$ and $f(y)=y^2-2y$ over the interval $[-1,2]$. So its area is

$$\int_{-1}^{2} \left[(4-y^2)-(y^2-2y)\right] dy$$